Problem with a solution proposed by Arkady Alt , San Jose , California, USA

For sequence $\{a_n\}_{n\geq 1}$ defined recursively by $a_{n+1}=\dfrac{a_n}{1+a_n^p}$ for $n\in\mathbb{N}$,

$a_1=a>0$,determine all positive real p for which series $\sum\limits_{n=1}^{\infty}a_n$ is convergent.

Solution.

First note that $a_n>0$ for all $n\in\mathbb{N}$ ($a_1=a>0$ and from supposition $a_n>0$ follows $a_{n+1}=\dfrac{a_n}{1+a_n^p}>0$.

Also note that sequence $\{a_n\}_{n\geq 1}$ is decreasing.

Indeed $a_n-a_{n+1}=a_n-\dfrac{a_n}{1+a_n^p}=\dfrac{a_n^{p+1}}{1+a_n^p}>0$.

Therefore, sequence $\{a_n\}_{n\geq 1}$ convergent to some nonnegative limit x.

Then $x=\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\dfrac{a_n}{1+a_n^p}=\dfrac{x}{1+x^p}\implies x=0$.

Thus, $\lim\limits_{n\to\infty}a_n=0$.

Since recurrence $a_{n+1}=\dfrac{a_n}{1+a_n^p}$ can be rewritten in the form

$$a_{n+1}^p=\frac{a_n^p}{(1+a_n^\alpha)^p},$$

then denoting a_n^p via b_n we obtain recurrence

(1) $\qquad b_{n+1}=\dfrac{b_n}{(1+b_n)^p}$,with initial condition $b_1=a^p$.

Since $\dfrac{1}{b_{n+1}}-\dfrac{1}{b_n}=\dfrac{(1+b_n)^p-1}{b_n}$ and $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}a_n^p=0$ then

$$\lim_{n\to\infty}\left(\frac{1}{b_{n+1}}-\frac{1}{b_n}\right)=\lim_{n\to\infty}\frac{(1+b_n)^p-1}{b_n}=p.$$

Hereof, by Arithmetic Mean Limit Theorem (if $\lim\limits_{n\to\infty}x_n=a$ then $\lim\limits_{n\to\infty}\dfrac{x_1+x_2+...+x_n}{n}=a$) we obtain

$$\lim_{n\to\infty}\frac{1}{nb_n}=\lim_{n\to\infty}\frac{\dfrac{1}{b_n}-\dfrac{1}{b_1}}{n-1}\cdot\frac{n-1}{n}=\lim_{n\to\infty}\frac{\sum\limits_{k=2}^{n}\left(\dfrac{1}{b_k}-\dfrac{1}{b_{k-1}}\right)}{n-1}=$$
$$\lim_{n\to\infty}\left(\frac{1}{b_n}-\frac{1}{b_{n-1}}\right)=p.$$

Thus, $\lim\limits_{n\to\infty}n^{\frac{1}{p}}a_n=\lim\limits_{n\to\infty}(na_n^p)^{\frac{1}{p}}=\lim\limits_{n\to\infty}(nb_n)^{\frac{1}{p}}=\left(\dfrac{1}{p}\right)^{\frac{1}{p}}$ and, therefore,

$$\lim_{n\to\infty}\frac{a_n}{\left(\dfrac{1}{np}\right)^{\frac{1}{p}}}a_n=1.$$

Hence, $\sum\limits_{n=1}^{\infty}a_n$ is convergent iff $\sum\limits_{n=1}^{\infty}\dfrac{1}{(np)^{\frac{1}{p}}}$ is convergent,that is iff $\dfrac{1}{p}>1\iff$ $p<1$.